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Attenuated wave-induced drift in a viscous rotating ocean

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Mean drift currents due to spatially periodic surface waves in a viscous rotating fluid are investigated theoretically. The analysis is based on the Lagrangian description of motion. The fluid is homogeneous, the depth is infinite, and there is no continuous energy input at the surface. Owing to viscosity the wave field and the associated mass transport will attenuate in time. For the non-rotating case the present approach yields the time-decaying Stokes drift in a slightly viscous ocean. The analysis shows that the drift velocities are finite everywhere. In a rotating fluid it is found that the effect of viscosity implies a non-zero net mass transport associated with the waves, as opposed to the result of no net transport obtained from inviscid theory (Ursell 1950).

1. Introduction

It was demonstrated theoretically by Stokes (1847) that small-amplitude surface waves induce a mean flow (mass transport) in the direction of wave propagation. Stokes' calculations are valid for a non-rotating inviscid fluid (irrotational motion). However, when applying Stokes theory to ocean waves, rotation must be taken into account. The most amenable case for theoretical studies is that of surface waves that have been generated by a distant agency. Such waves are known as swell. It is a well-known fact of observation that, at a sufficiently far distance from the generating (storm) area, they can reasonably well be treated as monochromatic wavetrains. For such waves Ursell (1950) demonstrated that in an *inviscid* ocean there could be no net steady mass transport. Similar conclusions were reached by Pollard (1970*a*) for a stratified rotating inviscid fluid.

The pertinent question to be raised is: what does the effect of viscosity do to the mass transport in a rotating ocean? Returning to the non-rotating case, it is worth recalling that the inclusion of a small viscosity into the problem produces some rather surprising results. In particular we refer to the papers by Longuet-Higgins (1953, 1960). Here he demonstrates that the inclusion of a small viscosity not only modifies the motion in thin boundary layers near the surface and bottom, but also produces significant changes from Stokes' solution in the interior! Most notably the mass transport velocity gradient just below the surface boundary layer was found to be *twice* the value obtained from Stokes solution.

Liu & Davis (1977) incorporated viscous attenuation into the problem (some of their results have been criticized by Dore 1978 and Craik 1982), and obtained a finite surface velocity when the depth increased towards infinity. This in contrast with Longuet-Higgins' (1953) apparent paradox of an infinite surface drift velocity for this limit (Huang 1970). The present paper discusses this point further.

The effect of rotation does limit the wave-induced mass transport to within an Ekman layer. For undamped surface waves (assuming an unspecified energy input from the wind) this was demonstrated by Madsen (1978) for a steady mean flow. However, when the ratio of the Ekman depth to the Stokes depth approaches infinity, the surface velocity in Madsen's analysis tends to infinity. A revised and corrected analysis of this problem has been performed by the author (Weber 1983). By prescribing the vertical wind-stress at the surface in such a way that the energy input from the wind exactly balances the energy loss in the wave motion due to viscous dissipation, a correct set of boundary conditions is obtained. The analysis then yields finite mass transports everywhere.

For tidal waves Lamoure and Mei (1977) demonstrate that the mass transport occurs in the bottom Ekman layer.

The present paper considers the attenuated swell problem in a rotating ocean, which is different from the wind-influenced case referred to above. The effect of viscosity yields results that differ from those obtained from inviscid theory (e.g. Ursell 1950). The purpose of this paper is to present these results together with their underlying assumptions.

We have chosen to use a Lagrangian description of motion. This turns out to be very convenient for the present problem. In particular, the second-order mean motion gives the mass transport velocity directly.

The mathematical formulation of the problem is given in §2. In §3 the first-order attenuated wave solution is given, and §4 yields the equations for the attenuated mean flow (mass transport). Since the non-rotating case has given rise to some controversy in the past, we devote §5 to the discussion of this problem. In §6 the full solution for a rotating fluid is presented. A summary together with a brief discussion of some further consequences of using a Lagrangian formulation are given in §7.

2. Mathematical formulation

The model of the problem is the same as in Weber (1983). To recapitulate briefly, we consider a homogeneous incompressible viscous fluid rotating about the vertical axis with a constant angular velocity $\frac{1}{2}f$, where f is the Coriolis parameter. The depth of the fluid is infinite, and the horizontal extent is unlimited. When undisturbed, the surface is horizontal. A Cartesian coordinate system is chosen such that the (x, y)-axes are situated at the undisturbed surface and the z-axis is positive upwards.

The motion is described by using a Lagrangian formulation. Let a fluid particle (a, b, c) initially have coordintes (x_0, y_0, z_0). Its position (x, y, z) at later times will then be a function of a, b, c and time t. Velocity components and accelerations are given by (x_t, y_t, z_t) and (x_{tt}, y_{tt}, z_{tt}), respectively, where subscripts denote partial differentiation. By including rotation, the equations for conservation of momentum and mass can be written (Lamb 1932; Pierson 1962) as

$$x_{tt}-fy_t = -\frac{1}{\rho_0}\frac{\partial(p,y,z)/\partial(a,b,c)}{\partial(x_0,y_0,z_0)/\partial(a,b,c)}+\nu\nabla^2 x_t, \tag{2.1}$$

$$y_{tt}+fx_t = -\frac{1}{\rho_0}\frac{\partial(x,p,z)/\partial(a,b,c)}{\partial(x_0,y_0,z_0)/\partial(a,b,c)}+\nu\nabla^2 y_t, \tag{2.2}$$

$$z_{tt}+g = -\frac{1}{\rho_0}\frac{\partial(x,y,p)/\partial(a,b,c)}{\partial(x_0,y_0,z_0)/\partial(a,b,c)}+\nu\nabla^2 z_t, \tag{2.3}$$

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\partial(x_0,y_0,z_0)}{\partial(a,b,c)}, \tag{2.4}$$

where p is the pressure, ρ_0 is the constant density, ν is the constant coefficient of kinematic viscosity and g the acceleration due to gravity. The operator connected with the pressure terms and the continuity equation is the Jacobian. The Laplacian operator ∇^2 becomes rather complicated when expressed in Lagrangian form, involving the Jacobian of a Jacobian. The reader is referred to Pierson (1962) for the explicit expression.

The equations (2.1)–(2.4) will be solved by considering small perturbations from the basic state (a, b, c). According to Pierson (1962) we write the solutions

$$\left.\begin{aligned} x &= a+\epsilon x^{(1)}+\epsilon^2 x^{(2)}+\ldots, \\ y &= b+\epsilon y^{(1)}+\epsilon^2 y^{(2)}+\ldots, \\ z &= c+\epsilon z^{(1)}+\epsilon^2 z^{(2)}+\ldots, \\ p &= -\rho_0 gc+\epsilon p^{(1)}+\epsilon^2 p^{(2)}+\ldots. \end{aligned}\right\} \tag{2.5}$$

Here we have put an arbitrary pressure term equal to zero. The expansion parameter ϵ is essentially proportional to the amplitude of the initial surface wave, as will be demonstrated later.

The free surface is given by $c = 0$ for all times. This is a linear kinematic boundary condition as opposed to the equivalent nonlinear Eulerian version. This shows some of the advantages by using a Lagrangian description for problems involving a freely moving boundary. Alternatively, in Eulerian form, one will have to use some sort of curvilinear coordinates along the free surface (Longuet-Higgins 1953).

The free surface is given by $z = \zeta$ in Eulerian form. By the aid of (2.5) the appropriate Lagrangian description of the surface form becomes

$$\zeta = \epsilon z^{(1)}+\epsilon^2 z^{(2)}+\ldots \quad (c = 0). \tag{2.6}$$

Since we work in an infinitely deep ocean, all perturbation quantities are assumed to vanish when $c \to -\infty$.

For the swell problem, the dynamic boundary condition at the free surface is that of vanishing normal and tangential stresses; see Ünlüata & Mei (1970) for explicit expressions to $O(\epsilon)$ and $O(\epsilon^2)$.

3. The attenuated primary wave

The linearized version of (2.1)–(2.4), or the solution to $O(\epsilon)$, yields the primary wave. We are looking at an (idealized) model of ocean swell, i.e. relatively short, high-frequency gravity waves. This means that the problem can be considerably simplified. Typically, such waves will have wavelengths λ of around 100 m and frequencies σ of about 1 s^{-1}. With a value of the inertial frequency $f \sim 10^{-4}$ s^{-1}, the Rossby radii of deformation for such waves are of the order 10^5 m. Hence we can safely neglect the effect of rotation on the solution to $O(\epsilon)$.

Letting the wave propagate along the x-axis ($\partial/\partial b = 0$ in the perturbations) and taking $y^{(1)} = 0$ since we neglect rotation, the first-order equations then reduce to

$$\left.\begin{aligned} x_{tt}^{(1)}+gz_a^{(1)} &= -\frac{1}{\rho_0}p_a^{(1)}+\nu\nabla_L^2 x_t^{(1)}, \\ z_{tt}^{(1)}+gz_c^{(1)} &= -\frac{1}{\rho_0}p_c^{(1)}+\nu\nabla_L^2 z_t^{(1)}, \\ x_a^{(1)}+z_c^{(1)} &= 0, \end{aligned}\right\} \tag{3.1}$$

where $\nabla_L^2 = \partial^2/\partial a^2 + \partial^2/\partial c^2$. We have here assumed that

$$\partial(x_0, y_0, z_0)/\partial(a, b, c) = 1 + O(\epsilon^2),$$

which will be verified later.

Formally (3.1) is identical with what one obtains from a linearized Eulerian description. Therefore the solution of (3.1) can be obtained directly from Lamb (1932, p. 625).

However, let us make one further simplification. Consider the viscous lengthscale γ^{-1} defined by

$$\gamma^{-1} = \left(\frac{2\nu}{\sigma}\right)^{\frac{1}{2}}, \tag{3.2}$$

which is commonly encountered in analyses of viscosity-influenced wave motion (e.g. Harrison 1909). Here $\sigma = (gk)^{\frac{1}{2}}$ is the frequency of small-amplitude deep-water waves with wavenumber k. The solution to $O(\epsilon)$ involves the ratio k/γ, and in what follows we shall assume that

$$k/\gamma \ll 1. \tag{3.3}$$

This is in fact seen to be very well fulfilled for the kind of waves we are looking at, even with a turbulent eddy viscosity in (3.2).

Introducing the attenuation coefficient β by

$$\beta = 2\nu k^2 = \frac{\sigma k^2}{\gamma^2}, \tag{3.4}$$

a set of normalized solutions can be written as

$$x^{(1)} = \frac{k}{\sigma} e^{-\beta t} \left\{ \left[e^{kc} - \frac{k}{\gamma} e^{\gamma c} (\cos \gamma c + \sin \gamma c) \right] \cos (ka - \sigma t) + \frac{k}{\gamma} \left[e^{\gamma c} (\cos \gamma c - \sin \gamma c) - \frac{k}{\gamma} e^{kc} \right] \sin (ka - \sigma t) + O\left(\frac{k^3}{\gamma^3}\right) \right\}, \tag{3.5}$$

$$z^{(1)} = \frac{k}{\sigma} e^{-\beta t} \left\{ \left[e^{kc} - \frac{k^2}{\gamma^2} e^{\gamma c} \sin \gamma c \right] \sin (ka - \sigma t) - \frac{k^2}{\gamma^2} \left[e^{\gamma c} \cos \gamma c - e^{kc} \right] \cos (ka - \sigma t) + O\left(\frac{k^4}{\gamma^4}\right) \right\}, \tag{3.6}$$

$$p^{(1)} = \rho_0 \sigma \frac{k^2}{\gamma^2} e^{-\beta t} \left\{ (e^{\gamma c} \cos \gamma c - 2e^{kc}) \cos (ka - \sigma t) + e^{\gamma c} \sin \gamma c \sin (ka - \sigma t) + O\left(\frac{k^2}{\gamma^2}\right) \right\}. \tag{3.7}$$

Here we have let the waves propagate along the positive x-axis. Normalized in the present context merely means that an arbitrary constant of order unity has been incorporated into the expansion parameter ϵ.

To this order the surface elevation ζ can be written from (2.6) as

$$\zeta = \epsilon z^{(1)}(c = 0) = \epsilon \frac{k}{\sigma} e^{-\beta t} \sin (ka - \sigma t). \tag{3.8}$$

Assume that we initially start out with a monochromatic wave $\zeta = \zeta_0 \sin (ka - \sigma t)$. By comparison with (3.8), we see that our expansion parameter can be written

$$\epsilon = \zeta_0 \sigma / k \tag{3.9}$$

(Weber 1983), or proportional to the wave amplitude as stated in §2.

From (3.5) and (3.6) we find for the initial volume of the fluid element in question that $\partial(x_0, y_0, z_0)/\partial(a, b, c) = 1+O(\epsilon^2)$, as previously assumed.

To a first approximation, neglecting the effect of viscosity, we obtain from (3.5) and (3.6) that a 'Fourier component' in Lagrangian formulation yields a trochoidal surface shape at a given instant. The effect of a small viscosity introduces a slight change towards a more symmetric form. In fact, for friction-dominated flow (creeping motion) the surface shape is purely sinusoidal (Tyvand & Weber 1983).

4. The attenuated secondary mean flow

For the equations to $O(\epsilon^2)$ we again refer to Pierson (1962). Averaging the second-order equations over one wavelength, and including rotation, we obtain for the horizontal mean flow

$$\begin{aligned} -\bar{x}^{(2)}_{tt} + f\bar{y}^{(2)}_t + \nu \bar{x}^{(2)}_{tcc} = & -\frac{1}{\rho_0}\overline{p^{(1)}_a x^{(1)}_a} - \frac{1}{\rho_0}\overline{p^{(1)}_c z^{(1)}_a} \\ & + \nu[2\overline{x^{(1)}_a x^{(1)}_{taa}} + 2\overline{z^{(1)}_c x^{(1)}_{tcc}} + 2\overline{z^{(1)}_a x^{(1)}_{tac}} \\ & + 2\overline{x^{(1)}_c x^{(1)}_{tac}} + \overline{x^{(1)}_{ta} \nabla^2_{\mathrm{L}} x^{(1)}} + \overline{x^{(1)}_{tc} \nabla^2_{\mathrm{L}} z^{(1)}}], \end{aligned} \tag{4.1}$$

$$-\bar{y}^{(2)}_{tt} - f\bar{x}^{(2)}_t + \nu \bar{y}^{(2)}_{tcc} = 0, \tag{4.2}$$

where the overbar denotes spatial average. We have here assumed that there is no mean horizontal pressure gradient to $O(\epsilon^2)$. The mean displacement to $O(\epsilon^2)$ of a particle is given by

$$(\bar{x}, \bar{y}, \bar{z}) = (a+\epsilon^2\bar{x}^{(2)}, b+\epsilon^2\bar{y}^{(2)}, c+\epsilon^2\bar{z}^{(2)}). \tag{4.3}$$

It is a direct measure of the net mass transport associated with the waves (see Andrews & McIntyre 1978; Weber 1983).

Following Weber (1983), we define horizontal mean-flow components u, v to $O(\epsilon^2)$ by

$$u = \epsilon^2 \bar{x}^{(2)}_t, \quad v = \epsilon^2 \bar{y}^{(2)}_t, \tag{4.4}$$

and a complex velocity W as

$$W = u + \mathrm{i}v. \tag{4.5}$$

Inserting for $x^{(1)}, z^{(1)}$ and $p^{(1)}$ from (3.5)–(3.7) into (4.1) and (4.2), and using the definitions above, the equation for the mean motion reduces to

$$\nu W_{cc} - W_t - \mathrm{i}fW = \nu\zeta_0^2\sigma k^3 \,\mathrm{e}^{-2\beta t}\left[8\mathrm{e}^{2kc} - \frac{4\gamma}{k}\mathrm{e}^{\gamma c}(\cos\gamma c - \sin\gamma c)\right]. \tag{4.6}$$

Here we again have utilized the fact that $\gamma \gg k$. This means that terms proportional to $\exp(\gamma c)$, $(k/\gamma)\exp(\gamma c)$, etc. have been neglected inside the square brackets on the right-hand side of (4.6). This can be done because, when integrating, one finds that these terms only introduce small corrections to the mean velocity or the mean-velocity gradient.

From Ünlüata & Mei (1970, equation (36)), the boundary condition to $O(\epsilon^2)$ in the horizontal direction can be written

$$W_c = -2\frac{\zeta_0^2\sigma^2}{k^2}\frac{\partial}{\partial t}\overline{(x^{(1)}_a x^{(1)}_c)} \quad (c = 0). \tag{4.7}$$

We notice that the right-hand side is proportional to k^2/γ^2. So, in the present limit of small k/γ, we have approximately

$$W_c = 0 \quad (c = 0). \tag{4.8}$$

In the case of no rotation, the condition (4.8) is equivalent to that obtained by Longuet-Higgins (1953, 1960) and Ünlüata & Mei (1970) for the Lagrangian mean velocity in the absence of viscous attenuation.

Furthermore, we must require

$$W \to 0 \quad (c \to -\infty). \tag{4.9}$$

5. The Stokes drift

First we consider the non-rotating case ($f = 0$, $v = 0$, $u = W$), as this subject has attracted the interest of so many writers since Stokes' analysis of the problem in 1847. As mentioned in §1 the inclusion of viscosity was shown to alter the mass transport drastically; also in the interior (Longuet-Higgins 1953). However, unless energy is transferred to the waves through the action of wind etc., the waves must attenuate due to friction. At first sight this effect might seem to be small, but, as demonstrated by Liu & Davis (1977), it should be included when the mass transport is considered.

In the non-rotating case (4.6) becomes

$$\nu u_{cc} - u_t = \nu \zeta_0^2 \sigma k^3 \, e^{-2\beta t} \left[8 e^{2kc} - \frac{4\gamma}{k} e^{\gamma c} (\cos \gamma c - \sin \gamma c) \right]. \tag{5.1}$$

We note that the form of the right-hand side depends crucially on the attenuation coefficient (3.4). If we, for example, attempt to neglect viscous attenuation in the computation of the first-order solution, the pressure $p^{(1)}$ from (3.7) would lack the term proportional to $-2(k^2/\gamma^2) \exp(kc)$. As a result, the square brackets on the right-hand side of (5.1) would contain a term $4 \exp(2kc)$ instead of $8 \exp(2kc)$. Secondly, when solving (5.1) the importance again becomes evident. For a solution of the form $\exp(-2\beta t + 2kc)$, we notice that the acceleration term u_t and the friction term νu_{cc} contribute equally to the solution, since $\beta = 2\nu k^2$.

Equation (5.1) is valid in the whole fluid, and determines the wave-induced drift to $O(\epsilon^2)$. To compare with the previous calculation of Longuet-Higgins (1953, 1960) and Ünlüata & Mei (1970) (hereinafter referred to as LH and Ü&M), we separate our fluid domain into a surface boundary-layer region and an interior. Since Ü&M work with a Lagrangian description, it is most convenient to compare with their results. For the boundary-layer region $0 \leqslant |c| \leqslant \gamma^{-1}$ the leading terms in (5.1) are

$$u_{cc} = -4\zeta_0^2 \sigma k^2 \gamma \, e^{\gamma c} (\cos \gamma c - \sin \gamma c) \, e^{-2\beta t}. \tag{5.2}$$

This is equal to Ü&M's equation (35), apart from the attenuation factor $\exp(-2\beta t)$. For the interior (5.1) reduces to

$$\nu u_{cc} - u_t = 8 \nu \zeta_0^2 \sigma k^3 \, e^{2kc - 2\beta t}. \tag{5.3}$$

If we had assumed $\beta = 0$ from the outset, the right-hand side of (5.3) would contain a factor 4 instead of 8. Neglecting the time dependence, (5.3) would then be equivalent to Ü&M's equation (45) for infinite depth and zero horizontal mean pressure gradient.

The boundary condition at the surface for the total mass transport velocity is given by (4.8), or

$$u_c = 0 \quad (c = 0), \tag{5.4}$$

which is the same as that of LH and Ü&M.

It is implicitly assumed in the works of LH and Ü&M that the wave field is maintained by some external device. By taking the steady mass-transport gradient to be zero at the surface, they find that the vorticity at the bottom of the surface layer becomes constant. Owing to diffusion into the interior, the mean horizontal momentum then will grow above all limits in an infinitely deep ocean.

The present paper considers a somewhat different approach. We assume that the wave field (3.5)–(3.7) is established at $t = 0$, and that no further energy is supplied. Then the energy and momentum will remain finite for all times. In fact, owing to viscous dissipation, the whole motion will finally tend to zero. It is therefore natural to assume that the boundary condition at infinity is given by (4.9), or

$$u \to 0 \quad (c \to -\infty). \tag{5.5}$$

A particular solution $u^{(p)}$ of (5.1) can be written

$$u^{(p)} = \zeta_0^2 \sigma k \left[e^{2kc} - \frac{2k}{\gamma} e^{\gamma c} (\sin \gamma c + \cos \gamma c) \right] e^{-2\beta t}. \tag{5.6}$$

Below the thin vorticity layer of thickness $\gamma^{-1} = (2\nu/\sigma)^{\frac{1}{2}}$ we note the interesting result that *our solution initially coincides with Stokes' inviscid solution.* It is obvious from (5.6) that, although the correction to the Stokes flow due to viscosity is small even at the surface (proportional to k/γ), the associated *gradient* at the surface is of order unity. This means that the particular solution $u^{(p)}$ does *not* fulfil the boundary condition (5.4). Accordingly, the complete solution must also include a solution $u^{(h)}$ of the homogeneous version of (5.1); that is,

$$\nu u_{cc}^{(h)} - u_t^{(h)} = 0, \tag{5.7}$$

subject to the boundary conditions

$$u_c^{(h)} = -u_c^{(p)} = 2\zeta_0^2 \sigma k^2 e^{-2\beta t} \quad (c = 0), \tag{5.8}$$

$$u^{(h)} \to 0 \quad (c \to -\infty). \tag{5.9}$$

The initial condition is

$$u^{(h)} = 0 \quad (t = 0). \tag{5.10}$$

The set (5.7)–(5.10) is usually solved by Laplace transforms, and this will be done in §6 for the full problem including rotation. We shall see that this diffusive solution is identical with that obtained by Longuet-Higgins (1969), introducing the concept of virtual wave stress at the surface.

It is appropriate here to make a few comments on Longuet-Higgins (1953, 1960) result of twice the Stokes gradient just below the surface boundary layer. From (5.2), with $\beta = 0$, we obtain

$$u_c = 4\zeta_0^2 \sigma k^2 (1 - e^{\gamma c} \cos \gamma c), \tag{5.11}$$

where we have used, as did L&H and Ü&M, the assumption that this particular solution satisfies $u_c = 0$ at $c = 0$. Hence, from (5.11),

$$u_c = 4\zeta_0^2 \sigma k^2, \tag{5.12}$$

when $|c| \gg \gamma^{-1}$, which is Longuet-Higgins' famous result. However, if we integrate (5.3) for the interior (with $\partial/\partial t = 0$, $\beta = 0$ and 4 instead of 8) and match the interior gradient with (5.12) in the limit $c \to 0$, further integration yields infinite mass-transport velocities in an infinitely deep ocean. This result is known as Longuet-Higgins' paradox.

The present way of looking at the problem avoids this difficulty since the particular solution (5.6) is finite at the surface. However, it does *not* fulfil the boundary condition $u_c = 0$ $(c = 0)$, and accordingly the second-order vorticity given by (5.8) must immediately start to diffuse inwards from the surface at $t = 0$. As mentioned before, Longuet-Higgins (1960, 1969) considered a similar diffusion of second-order mean vorticity from the bottom of the surface layer.

If the gradient (or flux) at $c = 0$ was constant, (5.7) would have the familiar solutions proportional to the integrated complementary error function of argument $c/2(\nu t)^{\frac{1}{2}}$. However, with a time-dependent boundary condition as in (5.8), the solution becomes more complicated. It might therefore be instructive to look at a simple approximate solution. Such a solution can be obtained by momentum/heat balance integral methods analogous to those of von Kármán (1921) and Pohlhausen (1921) for a viscous boundary layer, or Goodman (1958) for heat diffusion. In principle, the diffusion equation (5.7) is integrated over a viscous boundary-layer depth $\delta(t)$. The exact solution is approximated by a polynomial in c. The coefficients in the polynomial are determined by the boundary condition (5.8) and the additional conditions $u^{(h)} = u_c^{(h)} = u_{cc}^{(h)} = 0$ at $c = -\delta(t)$. Upon integration, this leads to a first-order differential equation for δ, subject to the initial condition $\delta(0) = 0$. The result is

$$u^{(h)} = \tfrac{2}{3}\zeta_0^2 \sigma k^2 \delta \left(1 + \frac{c}{\delta}\right)^3 \mathrm{e}^{-2\beta t}, \tag{5.13}$$

where

$$\delta = \frac{3^{\frac{1}{2}}}{k}(\mathrm{e}^{2\beta t} - 1)^{\frac{1}{2}}. \tag{5.14}$$

We should point out here that (5.13) is valid within the boundary layer, i.e. for $-\delta \leqslant c \leqslant 0$. Hence $u^{(h)}$ is always bounded. For small times the diffusion layer propagates into the fluid analogously to the constant-flux solution, i.e.

$$\delta \sim (12\nu t)^{\frac{1}{2}}. \tag{5.15}$$

If we form the fractional error $\Delta = (\nu u_{cc} - u_t)/\nu u_{cc}$, and insert values for the surface $c = 0$, we find $\Delta = \frac{1}{3}k^2\delta^2$. This error must be small, which shows that the approximate solution (5.13) is valid for small values of the dimensionless parameter $k\delta$.

The analysis above for the induced mean motion might apply to deep-water waves in a narrow channel, when there is no continuous energy input at the surface, and provided that the effects of endwalls can be neglected. However, for situations in the laboratory, these requirements clearly cannot be met. Here horizontal pressure gradients and/or return flow will result from the finite geometry of the model. Furthermore, a wavemaker must be operated in order to maintain the wave field. We shall therefore not attempt to compare our results with existing laboratory experiments (e.g. Russell & Osorio 1957; Longuet-Higgins 1960).

The main point of the present paper is to show that neither Stokes' classic solution (eventually modified by viscosity) nor existing laboratory experiments can be used for estimating wave drift associated with ocean swell. This is due to the importance of the joint action of viscosity and rotation, as will be demonstrated in §6.

6. The rotating ocean

We now return to the full problem including rotation. In addition to the Stokes depth and vorticity-layer depth, rotation introduces an Ekman depth into the problem. By definition

$$\left.\begin{aligned} L &= \frac{1}{2k} \quad \text{(Stokes depth)},\\ l &= \frac{1}{\gamma} = \left(\frac{2\nu}{\sigma}\right)^{\frac{1}{2}} \quad \text{(vorticity-layer depth)},\\ D &= \left(\frac{2\nu}{f}\right)^{\frac{1}{2}} \quad \text{(Ekman depth)}. \end{aligned}\right\} \tag{6.1}$$

The solution of (4.6) is obtained in the usual way by writing W as $W^{(p)}+W^{(h)}$, where $W^{(p)}$ is a particular solution of the inhomogeneous equation, and $W^{(h)}$ is a solution of the homogeneous equation.

Utilizing the fact that $l^2/D^2 = f/\sigma \ll 1$, the particular solution can be written

$$W^{(p)} = \zeta_0^2\sigma k\left[\frac{e^{2kc}}{1-iL^2/D^2}-\frac{2k}{\gamma}e^{\gamma c}(\cos\gamma c+\sin\gamma c)\right]e^{-2\beta t}. \quad (6.2)$$

This is just the attenuated Stokes drift which is deflected somewhat to the left (on the Northern Hemisphere) of the wave propagation direction. The rate of deflection is seen to depend on the ratio of the Stokes depth to the Ekman depth. Why this solution yields a flow component to the left, is easy to see. Consider flow in a rotating system in the presence of a pressure gradient. Then there is a tendency towards geostrophic balance in the regions where viscosity does not dominate. We note from (4.6) that, below the thin vorticity layer, the right-hand side is positive. This means that the mean flow experiences the forcing from the primary waves equivalent to an imposed pressure gradient. When we change sign in (4.6), we see that this corresponds to a force in the negative x-direction. Hence a tendency towards balance with the Coriolis force must induce a flow component in the positive y-direction, i.e. to the *left* of the wave propagation.

The solution of the homogeneous problem $W^{(h)}$ is obtained by Laplace transformation. The boundary conditions are

$$W_c^{(h)} = -W_c^{(p)} \quad (c=0), \quad (6.3)$$

$$W^{(h)}\to 0 \quad (c\to-\infty), \quad (6.4)$$

where $W^{(p)}$ is given by (6.2). The initial condition is

$$W^{(h)} = 0 \quad (t=0). \quad (6.5)$$

By using the shifting and convolution properties of Laplace transforms, we get

$$W^{(h)} = 2\zeta_0^2\sigma k^2\left(2-\frac{1}{1-iL^2/D^2}\right)\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}e^{-2\beta t}\int_0^t \frac{\exp\left[(2\beta-if)\xi-(c^2/4\nu\xi)\right]}{\xi^{\frac{1}{2}}}d\xi. \quad (6.6)$$

This solution describes the transient development of an Ekman current subject to a 'stress' (6.3) at the surface which decays exponentially in time.

By the substitution $r=-i\xi^{\frac{1}{2}}$ the integral in (6.6) can be expressed in terms of tabulated error functions for complex arguments (see Abramowitz & Stegun 1965, equations (7.1.3), (7.4.34) and table (7.9)).

Let us make a small digression to the non-rotating case. Putting $f=0$ in (6.6) and changing variables by $t_1 = t-\xi$, we obtain

$$u = 2\zeta_0^2\sigma k^2\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\int_0^t \frac{\exp\left[-2\beta t_1-c^2/4\nu(t-t_1)\right]}{(t-t_1)^{\frac{1}{2}}}dt. \quad (6.7)$$

This is exactly equal to Longuet-Higgins' (1969) solution (equation (4.6), p. 378) obtained by the concept of virtual tangential wave stress at the free surface. As pointed out before, this solution should be added to the Stokes flow (5.6) to yield the total mass transport in the interior of the fluid.

We return to the rotating ocean, and consider the surface flow. Initially we start

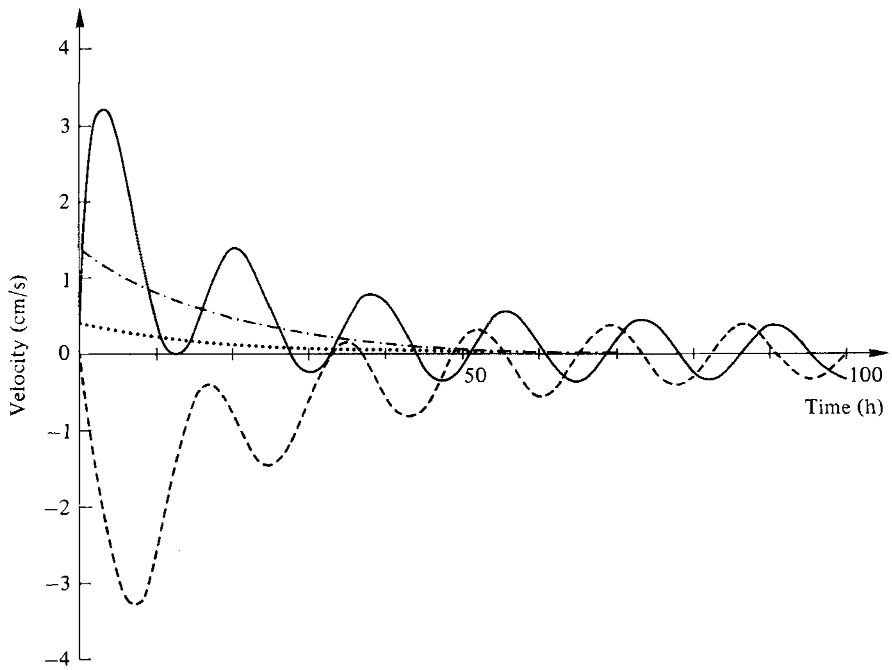

FIGURE 1. The various components of the mass transport velocity at the surface as function of time for $\nu = 10\ \mathrm{cm^2/s}$. The waves propagate along the x-axis with $\zeta_0 = 1$ m and $\lambda = 100$ m. ——, – – –, x- and y-components of the Ekman flow (6.6); · · ·, –·–, x- and y-components of the modified Stokes drift (6.2).

out with the (modified) Stokes solution (6.2). For small times the full solution at the surface can be written

$$\left.\begin{aligned} u &= \frac{\zeta_0^2\sigma k}{1+L^4/D^4}\left[1+\left(\frac{D^4+2L^4}{D^4}\right)\frac{D}{L}\left(\frac{2ft}{\pi}\right)^{\frac{1}{2}}\right]+O(t), \\ v &= \frac{\zeta_0^2\sigma k L^2/D^2}{1+L^4/D^4}\left[1-\frac{D}{L}\left(\frac{2ft}{\pi}\right)^{\frac{1}{2}}\right]+O(t). \end{aligned}\right\} \tag{6.8}$$

We note that rotation introduces the expected deflection of the velocity vector to the right of the initial flow direction.

The attenuation factor β in the solution cannot be arbitrarily chosen since it depends on the flow parameters. This means that the solution (6.6) cannot simply be compared with Fredholm's solution (Ekman 1905) by setting $\beta = 0$ as a particular example of constant wind forcing. In fact, if $\beta \rightarrow 0$, or equivalently $\nu \rightarrow 0$ when k is finite, then $L/D \rightarrow \infty$. Hence $W^{(\mathrm{p})} \rightarrow 0$ and $W^{(\mathrm{h})} \rightarrow 0$ in this case. This confirms Ursell's (1950) result that there is no net mass transport associated with ocean swell in a rotating inviscid ocean. However, the assumption of zero viscosity is not strictly valid for any fluid. The present paper demonstrates that even small viscosities result in a not negligible net mass transport.

In practice L and D could well be of the same order of magnitude in the ocean when relevant eddy viscosity coefficients are considered. A typical wavelength $\lambda = 100$ m gives a Stokes depth L of about 8 m. Take $\nu = 10\ \mathrm{cm^2/s}$, or about 10^3 times the molecular value. This is not an unreasonable value for the upper layer of the ocean. With $f = 10^{-4}\ \mathrm{s^{-1}}$, we get an Ekman depth D of about 4.5 m. In figure 1 we have displayed the different contributions to the mean current at the surface $c = 0$ for this

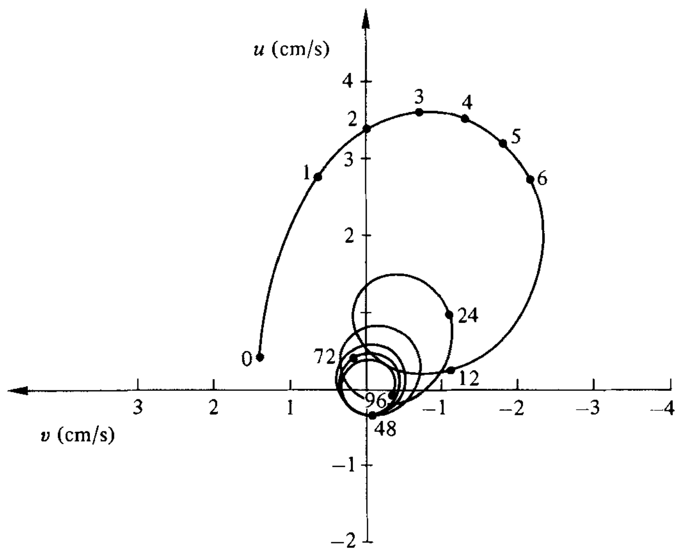

FIGURE 2. Hodograph of the total surface mass transport velocity $W^{(p)} + W^{(h)}$ from (6.2) and (6.6) for the same situation as in figure 1. The numbers denote time in hours after the onset of motion.

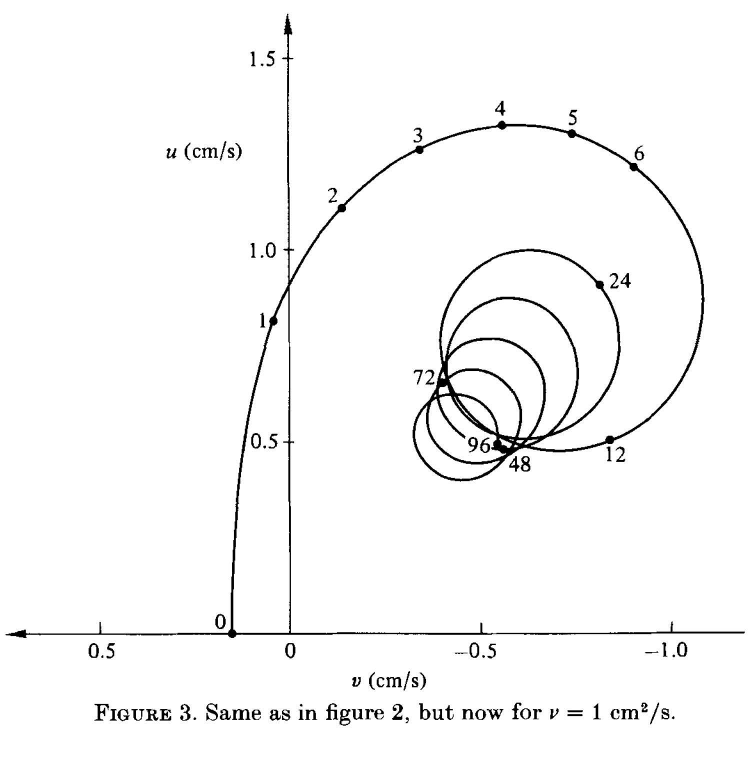

FIGURE 3. Same as in figure 2, but now for $\nu = 1$ cm^2/s.

particular example. For simplicity the integral in (6.6) was solved numerically. The initial amplitude of the wave travelling along the positive x-direction was taken to be 1 m. Apart from the expected flow component along the direction of wave propagation, we note that the modified Stokes flow has an additional component to the left of this direction, as explained in connection with the discussion of (6.2). The latter is in fact the larger in this example. It is interesting to note that the oscillatory

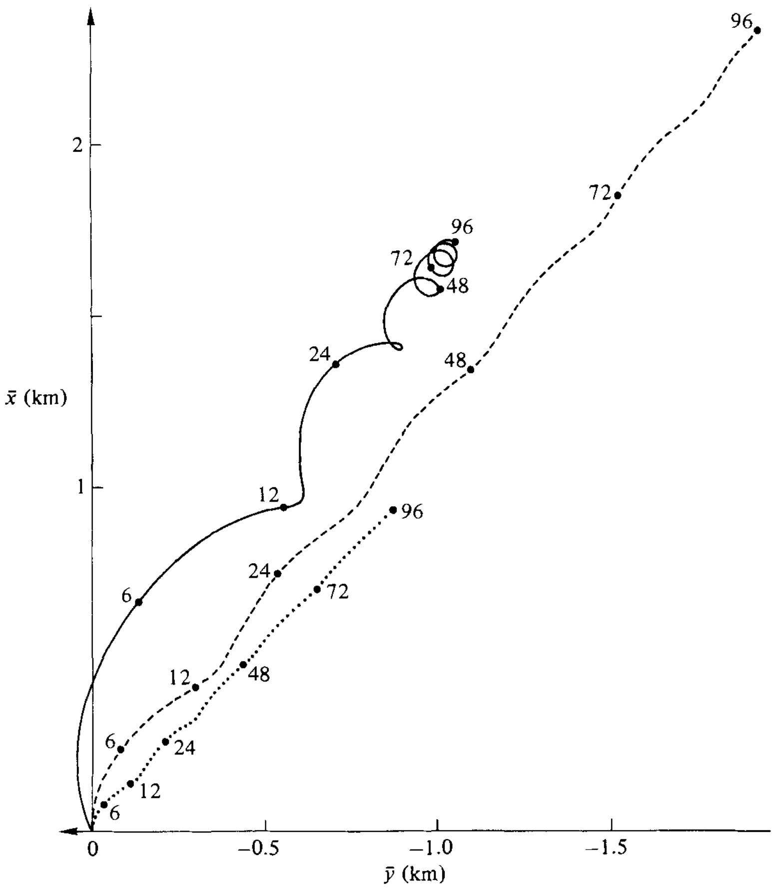

FIGURE 4. Horizontal mean displacements $\bar{x}, \bar{y}$ ($= \epsilon^2 \bar{x}^{(2)}, \epsilon^2 \bar{y}^{(2)}$) at the surface for various values of the viscosity coefficient. The waves propagate along x with $\zeta_0 = 1$ m and $\lambda = 100$ m. ——, $\nu = 10$ cm²/s; – – –, 1 cm²/s; · · ·, 0.1 cm²/s. Numbers denote time in hours after the onset of motion.

Ekman flow quickly becomes larger than the initial rotation-modified Stokes flow. When the forcing of the primary waves disappears (here after about 55 h), the induced motion continues for a long time as weakly damped inertial oscillations. The inertial period, here about $17\frac{1}{2}$ h, is easily detected from the graph.

In figure 2 we have plotted a hodograph of the total surface mass-transport velocity for the same situation as in figure 1. The numbers on the graph denote time in hours after the onset of motion. As mentioned before, we notice that the motion finally reduces to inertial oscillations. The radius of the approximate inertial circles decreases in time due to viscous dissipation, being about 40 m after 72 h.

When the viscosity becomes smaller, the viscous damping is less pronounced. But also the modified Stokes flow (6.2) and the induced Ekman current (6.6) become smaller. This is obvious from figure 3, where we have displayed the surface hodograph for the same case as before ($\lambda = 100$ m, $\zeta_0 = 1$ m), but with a viscosity $\nu = 1$ cm²/s. We note that, although the flow is small, it takes a very long time before it finally tends to zero. The computations were stopped after 96 h, and much more time is still needed before the spiralling hodograph ends up in the origin.

In figure 4 we have plotted the horizontal displacements associated with the second-order mean motion at the surface. The displacements are constructed from averaged velocities over periods of 15 min. With the particle definition of §4, the curves depicted in the graph are particle trajectories. The waves propagate along the x-axis with a wavelength of 100 m and an initial amplitude of 1 m. The numbers on the plot are time in hours after the onset of streaming motion, and the black dots denote position at specific times. The three curves correspond to eddy viscosities of 10, 1 and 0.1 cm^2/s, respectively. The first value might be reasonable for surface waters in not too windy conditions. The second represents an average value for vertical diffusion in the interior of the ocean (Munk 1966). It should therefore be adequate for calm surface conditions. The last value of 0.1 cm^2/s is probably on the lower side, but is kept for comparison.

It is interesting to observe how viscosity affects the net length of a trajectory travelled in a given time. Actually intermediate viscosities give the longest net displacements. This is also confirmed by additional computations for extremely high (10^2 cm^2/s) and extremely low (10^{-2} cm^2/s) values of ν (not displayed in the figure). The reason for this is that, although large viscosities result in initially large velocities, the motion becomes quickly damped in time. Small viscosities, on the other hand, give small velocities. Although the damping is slower, the resulting net displacements need not be very large.

It does not seem very sensible to continue the computations beyond 96 h. By that time a single wave phase would have travelled more than 4000 km, so it would probably have reached the shoreline by then.

We observe that the general effect of the Coriolis force is to deflect the particle motion to the right. On average, the surface trajectories plotted here are deflected about 40° to the right of the wave-propagation direction.

Results for lower levels ($c < 0$) have not been displayed, but as noticed from (6.2) and (6.6), the mean flow decreases exponentially with depth. At a given level the Ekman current (6.6) starts out with a lag in time compared to the surface value, and exhibits the familiar veering to the right.

We emphasize once again that the induced Ekman current may be several times larger than the initial modified Stokes current. However, as observed from the hodographs in figures 2 and 3, when the motion is finally reduced to inertial oscillations, the velocities have become quite small. Therefore it does not seem as if the existence of swell can explain the rather strong inertial currents reported by Pollard (1970*b*).

7. Summary and discussion

We have investigated theoretically mean drift currents due to spatially periodic surface waves (swell) in a homogeneous, deep, viscous, rotating ocean. The analysis is based on the Lagrangian description of motion. Perturbing the displacement field about a mean position, we obtain spatially averaged solutions to second order in a small parameter ϵ. This parameter is essentially proportional to the amplitude of the initial surface wave. There is no forcing from the wind. The considered waves have wavelengths which are small compared to the depth of the fluid, and their frequencies are much larger than the inertial frequency. Furthermore, the analysis assumes that $k/\gamma \equiv k(2\nu/\sigma)^{\frac{1}{2}} \ll 1$. Owing to viscosity, the waves, and also the induced secondary mean flow, become attenuated in time.

In the case of no rotation, the present way of looking at the problem yields finite drift velocities everywhere. If we apply the theory to ocean swell, the effect of rotation on the mean flow cannot be neglected. The inclusion of a non-zero viscosity leads to the important result that the associated mean flow has a non-zero mass transport. In the limit of vanishing viscosity, the mass transport tends to zero. This is in agreement with Ursell's (1950) result for swell in a rotating inviscid ocean.

We again emphasize that the use of a Lagrangian description of motion turns out to be very convenient for discussing viscous effects on wave-induced mean motion. Phillips (1977) argues that, because the equations of motion are singular to viscous perturbations, the streaming motion obtained with a small, but non-zero, viscosity does not tend to the proper inviscid limit when $\nu \rightarrow 0$. The present paper sheds some light on this problem. First, we take the non-rotating case, i.e. $f = 0$, $\bar{y}_t^{(2)} = 0$ in (4.1), (4.2). If now $\nu = 0$, then $p_a^{(1)} = p_c^{(1)} = 0$ from (3.7), and accordingly $\bar{x}_{tt}^{(2)} = 0$. Hence there is no coupling to the wave motion, and the mass-transport velocity will remain undetermined for all times. However, if we let $\nu \rightarrow 0$ (or $\gamma \rightarrow \infty$) in the result (5.6) which has been obtained with a small, but non-zero, ν, we regain Stokes' classic solution for the mass-transport velocity.

For the rotating case the situation is similar. Again the assumption of zero viscosity in (4.1), (4.2), results in a decoupling of the mean motion from the waves. The flow becomes undetermined in the same sense as before. But if there is any mean motion, it now must be purely inertial at any particular level of depth. This indefiniteness disappears when we let $\nu \rightarrow 0$ in the solution (6.2). We then obtain that the mass-transport velocity tends to zero everywhere, as it should according to inviscid theory. However, neither in the non-rotating nor in the rotating case does the mass transport gradient at the free surface $c = 0$ tend to the proper inviscid limit when $\nu \rightarrow 0$, exemplifying the singular nature of the problem.

The present study assumes the existence of monochromatic waves. In the ocean, however, the wave energy is distributed over a finite band of frequencies; although this can be quite narrow for ocean swell. The present theory also neglects interaction between waves and disintegration of wavetrains due to instability processes. Further, air effects are assumed to be negligible. Still it is hoped that this relatively simple analysis of an idealized situation may shed some additional light on the problem of induced mean motion due to surface waves.

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